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Univalent polynomials and fractional order differences of their coefficients ☆

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ABSTRACT

Using fractional order differences, we find sufficient conditions on the coefficients of certain polynomials defined on the unit disk $\mathbb{D}$: $\{z,\ |z|<1\}$ such that the zeros of these polynomials does not lie in $\mathbb{D}$.

1. Introduction

From a given sequence $\{a_k\}$, $k \geqslant 1$ of real or complex numbers, consider the backward difference operator ∇ and the right shift operator E defined by

$$\nabla a_k \equiv a_k - a_{k-1} \quad \text{and} \quad Ea_k = a_{k+1} \quad (a_{-1}=0), \tag{1.1}$$

respectively. Clearly the difference operator is linear. Thus, $\nabla = I - E^{-1}$ and symbolically, for a given complex number α, we can write

$$\nabla^\alpha = \left(I - E^{-1}\right)^\alpha = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} E^{-m},$$

where I is the identity operator given by $Ia_k = a_k$. Consequently, we write

$$\nabla^\alpha a_k = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} a_{k-m}. \tag{1.2}$$

If $a_{k-m}=0$ for $k-m<0$, then (1.2) leads to

$$\nabla^\alpha a_k = \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} a_{k-m}, \tag{1.3}$$

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or equivalently,

$$\nabla^{\alpha} a_k = (-1)^k \sum_{m=0}^{k} (-1)^m \binom{\alpha}{k-m} a_{k-m}.$$

Throughout this paper, by $\nabla^{\alpha} a_k$, we always mean the right-hand side of (1.3). The idea of using the difference operator for the present investigation is motivated from the work of G.T. Cargo and O. Shisha [3] where they discussed the construction of polynomials having no zeros within the unit disc $\mathbb{D}$. Let $\mathcal{A}$ denote the family of all analytic functions f defined in the unit disc $\mathbb{D} = \{z\colon\ |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.4}$$

Let

$$S^* = \{ f \in \mathcal{A}\colon\ f(\mathbb{D}) \text{ is a starlike domain with respect to the origin} \}.$$

A function $f \in \mathcal{A}$ is said to be *close-to-convex* with respect to a fixed starlike function $g \in S^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{1.5}$$

We are interested in the following

Problem 1.1. *Find conditions on* $\{a_n\}_{n \geqslant 1}$, $a_1 = 1$ *so that* f *defined by* (1.4) *is univalent in* $\mathbb{D}$.

Partial answer to the problem is well known in the literature (for example, see [9]). From (1.1) we find that

$$\nabla^2 a_k = \nabla(\nabla a_k) = \nabla(a_k - a_{k-1}) = a_k - 2a_{k-1} + a_{k-2}.$$

In view of this observation, the well-known result due to Fejér [6] takes the following form.

Lemma 1.1. *If* $\{a_n\}_{n \geqslant 1}$ *be a sequence of nonnegative real numbers such that* $a_1 = 1$, *and that the quantities* $\nabla n a_n$ *and* $\nabla^2 (n+1) a_{n+1}$ *are nonnegative for all* $n \geqslant 2$, *then* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *is starlike in* Δ.

Among others important results concerning univalency, results related to the difference operator ∇ that are useful for our discussion were due to Ozaki [10] and we give them in the following form.

Lemma 1.2. *(See [10].) Let* $f \in \mathcal{A}$ *be of the form* (1.4). *Suppose that any one of the following conditions holds*:

(i) $\displaystyle\sum_{n=2}^{\infty} |\nabla n a_n| \leqslant 1,$

(ii) $\displaystyle\sum_{n=2}^{\infty} \left|\nabla(1 + E^{-1}) n a_n\right| \leqslant 1,$

(iii) $\displaystyle\sum_{n=2}^{\infty} \left|\nabla^2 n a_n\right| \leqslant 1.$

Then $f(z)$ *is close-to-convex in* $\mathbb{D}$, *and hence univalent in* $\mathbb{D}$.

In particular, Lemma 1.2 gives the following: Let $f \in \mathcal{A}$ of the form (1.4) with nonnegative coefficients. Then, either of the conditions

$$\nabla k a_k \geqslant 0 \quad \text{for } k \geqslant 2 \quad \text{with } \lim_{n\to\infty} a_n \leqslant 2 \tag{1.6}$$

or

$$\nabla k a_k \leqslant 0 \quad \text{for all } k \leqslant 2, \tag{1.7}$$

is sufficient for f to be close-to-convex in $\mathbb{D}$. Similar conditions may be stated for functions satisfying one of the above conditions, given in Lemma 1.1.

We recall the following theorem due to G.T. Cargo and O. Shisha [3].

Theorem A. *(See [3].) Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ $(\not\equiv 0,\ n \geqslant 1)$ *be a polynomial, and let* $0 < \alpha \leqslant 1$. *Assume that* $a_k \geqslant 0$ $(k = 0, 1, \dots, n)$ *and that* $\nabla^\alpha a_k \leqslant 0$ $(k = 1, 2, \dots, n)$. *Then no zero of* $p(z)$ *lies in* $\mathbb{D}$.

Theorem A generalizes the work of Eneström [5], where Theorem A has been obtained only for the case $\alpha = 1$.

In this paper we obtain a number of results generalizing Lemma 1.2 and Theorem A with the operator ∇^α for $0 \leqslant \alpha \leqslant 2$. Since f is close-to-convex implies f is univalent [4, p. 45], we deduce univalency of f by obtaining close-to-convexity of f with respect to a starlike function.

2. Main results

Theorem 2.1. *Let* $f(z) = z + \sum_{k=2}^{n} a_k z^k$ $(\not\equiv 0)$ *be a polynomial, and* $0 \leqslant \alpha < 2$. *Assume that*

$$\sum_{k=2}^{\infty} \left|\nabla^\alpha k a_k\right| \leqslant 1.$$

Then f *is univalent in* $\mathbb{D}$.

Proof. Consider the function

$$(1-z)^\alpha = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} z^m, \quad 0 \leqslant \alpha < 2,\ z \in \mathbb{D}.$$

For convenience, we let $c_k = (k+1)a_{k+1}$ so that

$$(1-z)^\alpha f'(z) = \sum_{k=0}^{\infty} \nabla^\alpha c_k z^k = 1 + \sum_{k=1}^{\infty} \nabla^\alpha c_k z^k.$$

Thus,

$$\operatorname{Re}(1-z)^\alpha f'(z) \geqslant 1 - \sum_{k=1}^{\infty} \left|\nabla^\alpha (k+1) a_{k+1}\right| \geqslant 0, \quad z \in \mathbb{D},$$

and therefore, for $0 \leqslant \alpha < 2$, f is close-to-convex with respect to the starlike function $g(z) = z/(1-z)^\alpha$ which implies f is univalent in $\mathbb{D}$. $\square$

Theorem 2.2. *Let* $f(z) = z + \sum_{k=2}^{n} a_k z^k$ $(\not\equiv 0)$ *be a polynomial, and let* $1 \leqslant \alpha < 2$. *Assume that*

$$\left[I - \binom{\alpha}{1} E^{-1} + \binom{\alpha}{2} E^{-2}\right] (k+1)\, a_{k+1} > 0 \quad \text{for } k \geqslant 1. \tag{2.1}$$

Then f *is univalent in* $\mathbb{D}$.

Proof. Let $f(z)$ be given as in (1.4). Then, with $a_1 = 1$, we can write

$$\begin{aligned} (1-z)^\alpha f'(z) &= (1-z)^\alpha \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k \\ &= \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} (k-m+1) a_{k-m+1} \right\} z^k \\ &= \sum_{k=0}^{\infty} \nabla^\alpha (k+1) a_{k+1} z^k. \end{aligned} \tag{2.2}$$

Now, using $\sum_{k=0}^{\infty} \nabla^\alpha (k+1) a_{k+1} = 0$, we get

$$(1-z)^\alpha f'(z) = \sum_{k=1}^{\infty} \nabla^\alpha (k+1) a_{k+1} \left(z^k - 1\right).$$

Case 1: If $\alpha = 1$ then

$$\nabla (k+1) a_{k+1} = (k+1) a_{k+1} - k a_k$$

so that (2.2) can be written as

$$(1-z)f'(z)=1+\sum_{k=1}^{\infty}\nabla(k+1)a_{k+1}z^k.$$

Therefore,

$$\operatorname{Re}(1-z)f'(z)>1-\sum_{k=1}^{\infty}\left|\nabla(k+1)a_{k+1}\right|\geqslant 0.$$

If $\sum_{k=2}^{\infty}ka_k\leqslant 1$, we obtain the result from Lemma 1.2(i).

Case 2: We know that

$$\nabla^{\alpha}(k+1)a_{k+1}=(k+1)a_{k+1}-\alpha k a_k+\frac{\alpha(\alpha-1)}{2!}(k-1)a_{k-1}-\cdots$$
$$+(-1)^k\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}a_1.$$

Clearly, for $1<\alpha<2$ and $k>2$,

$$(-1)^k\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}(k-m+1)a_{k-m+1}>0.$$

Further, for $1<\alpha<2$, by the hypothesis of the theorem, we have

$$(k+1)a_{k+1}-\alpha k a_k+\frac{\alpha(\alpha-1)}{2!}(k-1)a_{k-1}>0,$$

which gives, for $z\in\mathbb{D}$,

$$\operatorname{Re}(1-z)^{\alpha}f'(z)=\operatorname{Re}\sum_{k=1}^{\infty}\nabla^{\alpha}(k+1)a_{k+1}\left(z^k-1\right)>0,$$

and the proof is complete. □

Remark 2.1. The range $0\leqslant\alpha\leqslant 1$ was given in [3]. We have taken $0\leqslant\alpha\leqslant 2$ (the range $\alpha=2$ is discussed in Theorem 2.3), due to the fact that the other ranges of α contain non-univalent starlike functions. For an interested reader on some extensions of these ranges, we refer to [12]. Further, the bound for α, given by $0\leqslant\alpha\leqslant 2$, is supported by the following result given in [8].

Theorem B. *(See [8].) $f\in\mathcal{S}^*$ if and only if there is a sequence of functions $\{f_n\}$ having the form*

$$g(z)=\frac{z}{\prod_{k=1}^{m}(1-x_kz)^{\lambda_k}}$$

where $|x_k|=1$, $\lambda_k\geqslant 0$ and $\sum_{k=1}^{m}\lambda_k=2$ and $f_n\to f$ uniformly on compact subsets of $\mathbb{D}$.

Now we consider the case $\alpha=2$. We have the following result.

Theorem 2.3. *Let $f\in\mathcal{A}$ be as in (1.4) and let f satisfy*

$$\nabla^2(k+1)a_{k+1}<0,\quad \text{for } k\geqslant 1,\quad \text{and}\quad a_1>2a_2.$$

Then, f is close-to-convex with respect to $g(z)=z/(1-z)^2$ in $\mathbb{D}$.

Proof. For $f\in\mathcal{A}$ of the form (1.4) we get from (1.3),

$$(1-z)^2f'(z)=(1-z)^2\left\{1+\sum_{k=1}^{\infty}(k+1)a_{k+1}z^k\right\}$$
$$=\sum_{k=1}^{\infty}\left(\nabla^2(k+1)a_{k+1}\right)\left\{z^k-1\right\}+2(a_1-2a_2).$$

Therefore, by the hypothesis given in the theorem, we have

$$\operatorname{Re}(1-z)^2f'(z)>0$$

and the proof is complete. □

Note that, we considered here $f \in \mathcal{A}$ to be in the form of (1.4), instead of a finite sum. This enable us to have the following remark.

Remark 2.2. The functions

$$z, \quad \frac{z}{1-z}, \quad \frac{z}{1-z^2}, \quad \frac{z}{(1-z)^2} \quad \text{and} \quad \frac{z}{1-z+z^2}$$

and their rotations are the only nine functions which are starlike univalent and have integer coefficients in $\mathbb{D}$, (see [7,11] for details). We note that, it is easy to give sufficient coefficient conditions for f to be close-to-convex, at least when the corresponding starlike function $g(z)$ takes one of the above forms. $g(z) = z$ is the well-known Nashiro–Warschowski theorem (see Duren [4, p. 47]). $g(z) = z/(1-z)$ is due to Eneström [5]. The case $g(z) = z/(1-z)^2$ is Theorem 2.3.

Hence we discuss the univalency for $f \in \mathcal{A}$ of the form (1.4) corresponding to the starlike functions $g(z) = z/(1-z^2)$ and $z/(1-z+z^2)$. The proof of Theorem 2.4 and Theorem 2.5 are similar to the one given for Theorem 2.3, and hence we only state the results.

Theorem 2.4. *Let $f \in \mathcal{A}$ be as in* (1.4) *and let f satisfy the following inequality*

$$\big(\nabla\big(1+E^{-1}\big)(k+1)a_{k+1}\big) < 0, \quad k \geqslant 1.$$

Then, f is close-to-convex with respect to $g(z) = z/(1-z^2)$ in $\mathbb{D}$.

Theorem 2.5. *Let $f \in \mathcal{A}$ be as in* (1.4) *and let, for $k \geqslant 1$, f satisfy*

$$a_k \leqslant 0 \quad \text{and} \quad \big(\big(1-E^{-1}+E^{-2}\big)(k+1)a_{k+1}\big) < 0.$$

Then, f is close-to-convex with respect to $g(z) = z/(1-z+z^2)$ in $\mathbb{D}$.

Finally, we give a better generalization of the above results in the following form.

Theorem 2.6. *Let $f \in \mathcal{A}$ be as in* (1.4) *and let f satisfy either*

$$a_k \geqslant 0 \quad \text{and} \quad \big(1+xE^{-1}\big)\big(1+yE^{-1}\big)(k+1)a_{k+1} < 0 \quad \text{for } k \geqslant 1,$$

together with $(1+x)(1+y) \geqslant 0$. Then, f is close-to-convex with respect to the function $g(z) = \frac{z}{(1+xz)(1+yz)}$, $|x| \leqslant 1$, $|y| \leqslant 1$ in $\mathbb{D}$.

Proof. In view of Theorems 2.3–2.5 and Remark 2.2, it is enough to prove the result for $|x| < 1$ and $|y| < 1$. For $f \in \mathcal{A}$ of the form (1.4) with $|x| < 1$ and $|y| < 1$, considering $(1+xz)(1+yz)f'(z)$ and proceeding as in previous theorems, we get the required result. $\square$

Note that in Theorem 2.6, $x = 1 = y$ gives Theorem 2.3 and $x = 1$, $y = -1$ gives Theorem 2.4. We cannot get such deduction for Theorem 2.5 from Theorem 2.6, as Theorem 2.5 is valid only for negative real coefficients.

3. Applications

We give some examples to support our results.

Example 3.1. Let $p_1(z) = 37 + 22z + 14z^2 + 11z^3 + 6z^4$, $z \in \mathbb{D}$. Then using the starlike function $g(z) = z/(1-z)^2$, by Theorem 2.3, $\nabla^2 c_k < 0$ and hence $p_1(z)$ has no zero inside $\mathbb{D}$. But the same conclusion about $\nabla^2 c_k$ cannot be obtained by using Theorem 2.4 and the starlike function $g(z) = z/(1-z^2)$.

Example 3.2. Let $p_2(z) = 15 + 8z + 3z^2 - 5z^3$, $z \in \mathbb{D}$. We observe from Theorem 2.4 that $c_k - c_{k-2} < 0$ and hence $\operatorname{Re}(1-z^2)p_2(z) \neq 0$ to conclude that $p_2(z)$ has no zeros inside $\mathbb{D}$. Similar conclusion cannot be obtained by considering $(1-z)^2 p_2(z)$ and using Theorem 2.3.

Example 3.3. Consider $p_3(z) = z + \alpha z^2 + \beta z^3$, $\alpha > 0$, $\beta > 0$. Then, by Theorem 2.3, we get $\alpha < \frac{1+3\beta}{2}$, which coincides with the result of Brannan [2] (see also [13, p. 78]). But, we observe that, we are not able to give any condition on β as given by these authors. In particular, when we take $\alpha = \sqrt{\frac{8}{9}}$ and $\beta = \frac{1}{3}$ [13, p. 79], we get that $p_3(z)$ is univalent.

Infact, it is easy to construct examples from the condition given in Theorem 2.2. For example, we find, the normalized Gaussian hypergeometric functions (details of which are given below), $zF(1,-2;2;z) = z - z^2 - \frac{1}{3}z^3$ satisfies (2.1) and hence univalent. Whereas, if $\{a_k\}$ are convex decreasing, then we cannot find condition on α so that (2.1) is satisfied. Hence, Theorem 2.2 is not useful in constructing polynomials that have convex decreasing coefficients.

Now, we are interested in finding conditions on the triplet a, b and c so that $zF(a,b;c;z)$ is close-to-convex with respect to any of the given starlike functions, where $F(a,b;c;z)$ is the well-known Gaussian hypergeometric function given by

$$F(a,b;c;z) := {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (n \in \mathbb{N}).$$

Here a, b and c are complex numbers with $c \neq 0, -1, -2, \ldots,$ and in the case of $c = -k$, $k = 0, 1, 2, \ldots,$ $F(a,b;c;z)$ is defined if $a = -j$ or $b = -j$ where $j \leqslant k$ and hence $F(a,b;c;z)$ becomes a polynomial of degree j in z.

As we are considering the normalized Gaussian hypergeometric function, we have

$$zF(a,b;c;z) = z + \sum_{n=2} A_n z^n, \quad z \in \mathbb{D}, \tag{3.1}$$

where

$$A_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}, \quad n = 2, 3, 4, \ldots.$$

The close-to-convexity of $zF(a,b;c;z)$ with respect to the starlike function $g(z) = z$ and $g(z) = z/(1-z)$ has been considered by various authors and partial results were obtained, see for example [1,14] and references therein. Hence we restrict our interest to the remaining cases.

Theorem 3.1. *Let a, b, c be real such that $(a-1)(b-1) < 0$ with $ab > 1$ and $a+b > 1/2$. Further, let a, b and c satisfy the following conditions*: $c \geqslant \max\{0, a+b\}$,

$$c^2 + c - (a+b)(a+b+1) + 2(1-ab) \geqslant 0, \tag{3.2}$$

$$c^2 - c - 2ab(a+b+1) - (a+b)(a+b+5) - 2(ab+1) \geqslant 0. \tag{3.3}$$

Then the hypergeometric function $zF(a,b;c;z)$ given by (3.1) *is close-to-convex with respect to the starlike function $z/(1-z^2)$.*

Proof. Consider $zF(a,b;c;z)$ given by (3.1). Then by Theorem 2.4, it suffices to show that

$$(k-1)A_{k-1} - (k+1)A_{k+1} \geqslant 0 \quad \text{for } k \geqslant 1. \tag{3.4}$$

A simple computation shows that (3.4) is equivalent to

$$\frac{(a)_{k-2}\,(b)_{k-2}}{(c)_k(1)_k} T(a,b,c) \geqslant 0$$

for all $k \geqslant 1$, where

$$T(a,b,c) = 2(c-a-b)(k-1)^4 + A_3(k-1)^3 + A_2(k-1)^2 + A_1(k-1) + 2ab(a-1)(1-b),$$

with

$$A_3 = c^2 + c - a^2 - b^2 - a - b - 4ab + 2,$$

$$A_2 = c^2 - c - 2a^2b - 2ab^2 - 4ab - a^2 - b^2 + 5b + 5a - 2, \quad \text{and}$$

$$A_1 = 2a^2 + 2b^2 - 3a^2b - 3b^2a - a^2b^2 - 2a - 2b + 7ab.$$

The proof is complete, if we show each of the coefficients of powers of $(k-1)$ are non-negative. By the hypothesis, using $c \geqslant a+b$, we get the coefficient of $(k-1)^4$ is non-negative. Again by using the conditions $(a-1)(b-1) < 0$ with $ab > 1$ and $a+b > 1/2$, we get that $A_1 \geqslant 0$ and the term $2ab(a-1)(b-1) \geqslant 0$. Now, using (3.2) and (3.3), we get that A_3 and A_4 are non-negative which implies $T(a,b,c) \geqslant 0$ and the proof is complete. □

We note that similar result cannot be obtained in the same method for $c = a+b$, whereas when $b = 1$, we get a special case and we give this as a corollary.

Corollary 3.1. *Let a, c be real such that $a > 1$. Further, let a and c satisfy the following conditions*: $c \geqslant \max\{0, a+1\}$,

$$c^2 + c - a^2 + a + 3 \geqslant 0, \tag{3.5}$$

$$c^2 - c - 2a(a+2) - (a+1)(a+8) \geqslant 0. \tag{3.6}$$

Then the incomplete beta function $\Phi(a; c; z)$ given by

$$\Phi(a; c; z) := zF(a, 1; c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n$$

is close-to-convex with respect to the starlike function $z/(1-z^2)$.

We are interested in finding conditions for the close-to-convexity of $zF(a, b; c; z)$ with respect to the starlike function $z/(1-z)^2$. We employ the same technique as in Theorem 3.1 and use the condition given in Theorem 2.3, to get the range for a, b and c, and we state it as a theorem without proof, as it follows a similar procedure given in the proof of Theorem 3.1.

Theorem 3.2. *Let a, b, c be real and $c \geqslant 2ab$. Further, let*

$$c \geqslant \max\left\{0,\ a+b+1,\ a+b+\frac{1-ab}{1+ab},\ a+b+\frac{ab}{2}+1+\frac{1}{2}\sqrt{8ab+a^2b^2}\right\} \tag{3.7}$$

together with

$$c \geqslant ab + \frac{1-(1-a)(1-b)}{1+(1-a)(1-b)}.$$

Then, the normalized Gaussian hypergeometric function $zF(a, b; c; z)$ is close-to-convex with respect to $z/(1-z)^2$ in $\mathbb{D}$.

Note that for the case $b = 1$, Theorem 3.2 cannot reduce to a corollary similar to Corollary 3.1 as this will not satisfy the hypothesis of the Theorem 3.2. But for $c = a + b$, we get the conditions given in Theorem 3.2 reduces to the following corollary.

Corollary 3.2. *Let $0 \leqslant 2ab \leqslant a + b \leqslant 1$, then $zF(a, b; a + b; z)$ is close-to-convex with respect to $z/(1-z)^2$.*

For the starlike functions $g(z) = z/(1-z^2)$ and $g(z) = z/(1-z)^2$ conditions obtained in Theorems 3.1 and 3.2 are better than many other results available in the literature. We refer to [1] for details. It is not possible to find similar conditions for the function $g(z) = z/(1-z+z^2)$ by employing the technique used in Theorems 3.1 and 3.2 as Theorem 2.5 involves negative coefficients. No result in this direction seem to be available in the literature. Hence it will be interesting to see if one can find applications in this direction. Note that, when we take a or b as a negative integer, due to the fact that the resulting polynomial will have real coefficients with alternating sign, our results are not applicable to this situation. Hence it will be worth to study further in this direction.

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